AIAA JOURNAL
Vol. 41, No. 6, June 2003

Geometrically Exact, Intrinsic Theory for Dynamics of Curved and Twisted Anisotropic Beams

Dewey H. Hodges*
Georgia Institute of Technology, Atlanta, Georgia 30332-0150

A formulation is presented for the nonlinear dynamics of initially curved and twisted anisotropic beams. When the applied loads at the ends of, and distributed along, the beam are independent of the deformation, neither displacement nor rotation variables appear: an "intrinsic" formulation. Like well-known special cases of these equations governing nonlinear dynamics of rigid bodies and nonlinear statics of beams, the complete set of intrinsic equations has a maximum degree of nonlinearity equal to two. Advantages of such a formulation are demonstrated with a simple example. When the initial curvature and twist are constant along the beam, two space–time conservation laws are shown to exist, one being a work–energy relation and the other a generalized impulse–momentum relation. These laws can be used, for example, as benchmarks to check the accuracy of any proposed solution, including time-marching and finite element schemes. The structure of the intrinsic equations suggests parallel approaches to spatial and temporal discretization. A particularly simple spatial discretization scheme is presented for the special case of the nonlinear static behavior of end-loaded beams that, by virtue of the Kirchhoff analogy, leads to a time-marching scheme for the dynamics of a pivoted rigid body in a gravity field. This time-marching scheme conserves both the angular momentum about a vertical line passing through the pivot and total mechanical energy, whereas the analogous spatial discretization scheme for the nonlinear static behavior of end-loaded beams satisfies analogous integrals of deformation along the beam span. Remarkably, a straightforward generalization of these discretization schemes is shown to satisfy both space–time conservation laws for the nonlinear dynamics of beams when the applied loads are constant within a space–time element.

Nomenclature

$\boldsymbol{B}_i$	=	unit vectors fixed in cross-sectional frame of deformed beam
$\boldsymbol{b}_i$	=	unit vectors fixed in cross-sectional frame of undeformed beam
C	=	direction cosine matrix with elements C_{ij}
C_{ij}	=	$\boldsymbol{B}_i \cdot \boldsymbol{b}_j$
e_1	=	$\lfloor 1 \ \ 0 \ \ 0 \rfloor^T$
F	=	column matrix with elements F_i
$\boldsymbol{F}$	=	cross-sectional stress resultant force vector
F_i	=	$\boldsymbol{F} \cdot \boldsymbol{B}_i$
f	=	column matrix with elements f_i
$\boldsymbol{f}$	=	distributed applied force vector per unit length along beam
f_i	=	$\boldsymbol{f} \cdot \boldsymbol{B}_i$
H	=	column matrix with elements H_i
$\boldsymbol{H}$	=	cross-sectional inertial angular momentum vector
H_i	=	$\boldsymbol{H} \cdot \boldsymbol{B}_i$
I	=	cross-sectional (3×3) inertia matrix
i_2, i_3, i_{23}	=	cross-sectional mass moments and product of inertia
K	=	column matrix with elements K_i
$\boldsymbol{K}$	=	deformed beam curvature and twist vector
K_i	=	$\boldsymbol{K} \cdot \boldsymbol{B}_i$
k	=	column matrix with elements k_i
$\boldsymbol{k}$	=	undeformed beam curvature and twist vector
k_i	=	$\boldsymbol{k} \cdot \boldsymbol{b}_i$
M	=	column matrix with elements M_i
$\boldsymbol{M}$	=	cross-sectional stress resultant moment vector
M_i	=	$\boldsymbol{M} \cdot \boldsymbol{B}_i$
m	=	column matrix with elements m_i
$\boldsymbol{m}$	=	distributed applied moment vector per unit length along beam
m_i	=	$\boldsymbol{m} \cdot \boldsymbol{B}_i$
P	=	column matrix with elements P_i
$\boldsymbol{P}$	=	vector of cross-sectional inertial linear momentum
P_i	=	$\boldsymbol{P} \cdot \boldsymbol{B}_i$
R, S, T	=	cross-sectional (3×3) flexibility coefficient matrices; Eqs. (3) and (4)
$\mathcal{T}$	=	kinetic energy per unit length
t_1, t_2	=	arbitrary instants in time
$\mathcal{U}$	=	strain energy per unit length
u	=	column matrix with elements u_i
$\boldsymbol{u}$	=	displacement vector of reference line
u_i	=	$\boldsymbol{u} \cdot \boldsymbol{b}_i$
V	=	column matrix with elements V_i
$\boldsymbol{V}$	=	inertial velocity vector of reference line
V_i	=	$\boldsymbol{V} \cdot \boldsymbol{B}_i$
$\bar{x}$	=	$\lfloor 0 \ \ \bar{x}_2 \ \ \bar{x}_3 \rfloor^T$
$\bar{x}_2, \bar{x}_3$	=	position coordinates along $\boldsymbol{b}_2, \boldsymbol{b}_3$ from reference line to cross-sectional mass centroid
γ	=	$\lfloor \gamma_{11} \ \ 2\gamma_{12} \ \ 2\gamma_{13} \rfloor^T$
γ_{11}	=	extensional strain of the reference line
$2\gamma_{12}, 2\gamma_{13}$	=	transverse shear strain measures of the reference line
Δ	=	identity matrix (3×3)
κ	=	column matrix with elements κ_i
κ_i	=	$K_i - k_i$
μ	=	mass per unit length
Ω	=	column matrix with elements Ω_i
$\boldsymbol{\Omega}$	=	inertial angular velocity vector of deformed beam cross-sectional frame
Ω_i	=	$\boldsymbol{\Omega} \cdot \boldsymbol{B}_i$

Superscripts

$\dot{}$	=	$\partial(\ )/\partial t$
$'$	=	$\partial(\ )/\partial x_1$
$\sim$	=	antisymmetric 3×3 matrix associated with a column matrix; Eq. (2)

Received 28 June 2002; revision received 12 January 2003; accepted for publication 22 January 2003.

*Professor, School of Aerospace Engineering; Dewey.Hodges@AE.GaTech.edu. Fellow AIAA.

Introduction

THE intrinsic form of the geometrically exact, nonlinear equations of equilibrium for beams has been established for well over a century, going back to Kirchhoff and Clebsch (see Ref. 1). Their classical formulation, which includes extension, twist, and bending, was extended to include transverse shear deformation by Reissner.[2] The equations of motion for the dynamics of beams appear to have first been written in intrinsic form in Ref. 3. These equations were solved in displacement form. Later, a mixed formulation was created based on the intrinsic motion and constitutive equations, both spatial and temporal,[4] with spatial and temporal kinematic equations adjoined by Lagrange multipliers. This formulation has been used, for example, in the analysis of fixed-wing aeroelasticity,[5,6] as well as helicopter rotor dynamics and aeroelasticity.[7,8] The advantages include low-order nonlinearities in the equations of motion and the absence of displacement and finite rotation variables from the equations of motion. Still, the adjoined kinematical equations must contain displacement and finite rotation variables to be used. Although one may find that having such variables appear only in a subset of the equations is an attractive feature, the kinematical equations must contain infinite degree nonlinearities unless Euler parameters or the direction cosines themselves are used as finite rotation variables. These latter approaches add more rotational variables and more Lagrange multipliers, thus creating more unknowns. Finally, the high-order nonlinearities and/or additional unknowns inherent in these approaches make analytical solutions more laborious.

This paper presents an alternative approach, one in which displacement and finite rotation variables do not appear. Of course, they or a subset of them can be added to the formulation to make possible their recovery; even so, depending on the problem, it may be possible to avoid nonlinearities of order greater than two and still have fewer equations and unknowns than would be necessary for the usual approach applied to the most general case.

The paper is organized in the following way. First, the intrinsic formulation for beams is reviewed, including the equations of motion, the spatial and temporal constitutive equations, and the spatial and temporal kinematical equations needed to close the formulation. From a special case of the kinematical equations used in Ref. 4, the intrinsic kinematical equations are derived. Advantages of this formulation are demonstrated with a simple example. The equations are then used to obtain two space–time conservation laws. The remainder of the paper deals with discretization schemes. A simple discretization scheme for the nonlinear statics of an end-loaded beam is presented that, by virtue of the Kirchhoff analogy, gives rise to a time-marching scheme for the dynamics of a rigid body. The relationship of these schemes to integrals of the motion or deformation is then discussed. Finally, these discretization schemes are generalized to form a space–time discretization scheme, the relationship of which to the space–time conservation laws is presented.

Intrinsic Formulation for Beams

Equations of Motion

Consider an initially curved and twisted beam of length ℓ undergoing finite deformation as shown in Fig. 1. The displacement vector beam of the reference line is denoted by $\boldsymbol{u}(x_1, t)$, where x_1 is the running length coordinate along the undeformed beam axis of cross-sectional centroids $\boldsymbol{r}$. The orthogonal set of basis vectors for the cross section of the undeformed beam is denoted by $\boldsymbol{b}_i(x_1)$, where $\boldsymbol{b}_1$ is chosen to be tangent to the reference line. The orthogonal set of basis vectors for the cross section of the deformed beam is denoted by $\boldsymbol{B}_i(x_1, t)$, where $\boldsymbol{B}_1$ is not in general tangent to the reference line of the deformed beam $\boldsymbol{R}$. The direction cosines are denoted as $C_{ij}(x_1, t) = \boldsymbol{B}_i \cdot \boldsymbol{b}_j$. Note that when the beam is in its undeformed state these two sets of unit vectors coincide, so that C reduces to the identity matrix. The warping of the cross section, both in and out of its original plane, is taken into account in the calculation of the cross-sectional constitutive law, discussed hereafter, such that the warping displacement is not constrainable at the boundaries. This

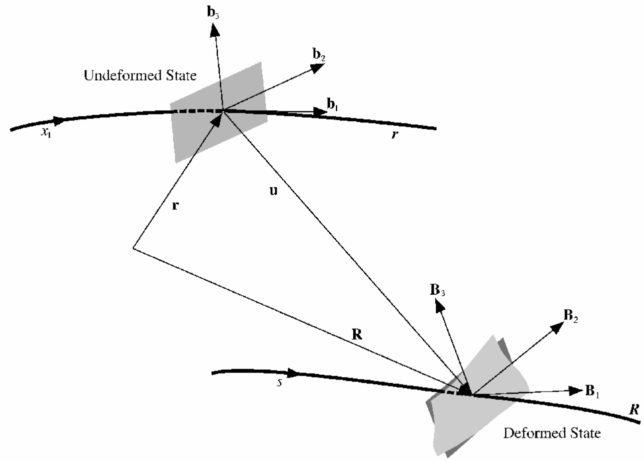

Fig. 1 Schematic of initially curved and twisted beam undergoing finite deformation, including cross-sectional warping.

constitutive law is, thus, suitable only for beams with closed cross sections.

The equations of motion in matrix form, as derived in Ref. 4, are given by

$$F' + \tilde{K}F + f = \dot{P} + \tilde{\Omega}P$$
$$M' + \tilde{K}M + (\tilde{e}_1 + \tilde{\gamma})F + m = \dot{H} + \tilde{\Omega}H + \tilde{V}P \qquad (1)$$

where the prime denotes the partial derivative with respect to the axial coordinate x_1 and the overdot denotes the partial derivative with respect to the time t; column matrices $F = \lfloor F_1 \; F_2 \; F_3 \rfloor^T$, $M = \lfloor M_1 \; M_2 \; M_3 \rfloor^T$, $\gamma = \lfloor \gamma_{11} \; 2\gamma_{12} \; 2\gamma_{13} \rfloor^T$, $K = \lfloor K_1 \; K_2 \; K_3 \rfloor^T$, $P = \lfloor P_1 \; P_2 \; P_3 \rfloor^T$, $H = \lfloor H_1 \; H_2 \; H_3 \rfloor^T$, $V = \lfloor V_1 \; V_2 \; V_3 \rfloor^T$, and $\Omega = \lfloor \Omega_1 \; \Omega_2 \; \Omega_3 \rfloor^T$; $f = \lfloor f_1 \; f_2 \; f_3 \rfloor^T$ and $m = \lfloor m_1 \; m_2 \; m_3 \rfloor^T$; γ_{11} is the extensional strain measure, and $2\gamma_{12}$ and $2\gamma_{13}$ are the transverse shear measures; and, finally, the tilde over certain terms denotes an antisymmetric matrix of the components of the column matrix of the same name, namely,

$$\tilde{K} = \begin{bmatrix} 0 & -K_3 & K_2 \\ K_3 & 0 & -K_1 \\ -K_2 & K_1 & 0 \end{bmatrix} \qquad (2)$$

All of the unknowns are functions of x_1 and t. It is helpful to relate the various column matrices to their associated vector quantities and the cross-sectional basis vectors of the undeformed and deformed beam. The various indexed scalar variables have the following meanings: $F_i = \boldsymbol{F} \cdot \boldsymbol{B}_i$ with $\boldsymbol{F}(x_1, t)$ being the resultant force of all tractions on the cross-sectional face at a particular value of x_1 along the reference line, $M_i = \boldsymbol{M} \cdot \boldsymbol{B}_i$ with $\boldsymbol{M}(x_1, t)$ being the resultant moment about the reference line at a particular value of x_1 of all tractions on the cross-sectional face, $K_i = \boldsymbol{K} \cdot \boldsymbol{B}_i$ with $\boldsymbol{K}(x_1, t)$ being the curvature of the deformed beam reference line at a particular value of x_1 such that $\boldsymbol{B}_i' = \boldsymbol{K} \times \boldsymbol{B}_i$, $V_i = \boldsymbol{V} \cdot \boldsymbol{B}_i$ with $\boldsymbol{V}(x_1, t)$ being the inertial velocity of a point at a particular value of x_1 on the deformed beam reference line, $\Omega_i = \boldsymbol{\Omega} \cdot \boldsymbol{B}_i$ with $\boldsymbol{\Omega}(x_1, t)$ being the inertial angular velocity of the deformed beam cross-sectional frame such that $\dot{\boldsymbol{B}}_i = \boldsymbol{\Omega} \times \boldsymbol{B}_i$, $P_i = \boldsymbol{P} \cdot \boldsymbol{B}_i$ with $\boldsymbol{P}(x_1, t)$ being the inertial linear momentum of the material points that make up the deformed beam reference cross section at a particular value of x_1, $H_i = \boldsymbol{H} \cdot \boldsymbol{B}_i$ with $\boldsymbol{H}(x_1, t)$ being the inertial angular momentum of all of the material points that make up a reference cross section of the deformed beam about the reference line of that cross section at a particular value of x_1, $f_i = \boldsymbol{f} \cdot \boldsymbol{B}_i$ with $\boldsymbol{f}(x_1, t)$ being the applied distributed force per unit length, and $m_i = \boldsymbol{m} \cdot \boldsymbol{B}_i$ with $\boldsymbol{m}(x_1, t)$ being the applied distributed moment per unit length. More details are given in Ref. 4.

Constitutive Equations

It is not necessary to retain all of the variables in Eq. (1). For the purposes of the present discussion, the generalized strains and momenta will be eliminated. For small strain, constitutive equations implied in Ref. 4 are linear and are here written in the form

$$\begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & S_{11} & S_{12} & S_{13} \\ R_{12} & R_{22} & R_{23} & S_{21} & S_{22} & S_{23} \\ R_{13} & R_{23} & R_{33} & S_{31} & S_{32} & S_{33} \\ S_{11} & S_{21} & S_{31} & T_{11} & T_{12} & T_{13} \\ S_{12} & S_{22} & S_{32} & T_{12} & T_{22} & T_{23} \\ S_{13} & S_{23} & S_{33} & T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{3}$$

The coefficients $R_{11}, R_{12}, \ldots, T_{33}$ are cross-sectional flexibility coefficients that can be obtained from a variety of means.[9,10] This equation may also be written as

$$\begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \begin{bmatrix} R & S \\ S^T & T \end{bmatrix} \begin{Bmatrix} F \\ M \end{Bmatrix} \tag{4}$$

Similarly, the generalized momentum–velocity relations[4] are

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ H_1 \\ H_2 \\ H_3 \end{Bmatrix} = \begin{bmatrix} \mu & 0 & 0 & 0 & \mu\bar{x}_3 & -\mu\bar{x}_2 \\ 0 & \mu & 0 & -\mu\bar{x}_3 & 0 & 0 \\ 0 & 0 & \mu & \mu\bar{x}_2 & 0 & 0 \\ 0 & -\mu\bar{x}_3 & \mu\bar{x}_2 & i_2 + i_3 & 0 & 0 \\ \mu\bar{x}_3 & 0 & 0 & 0 & i_2 & i_{23} \\ -\mu\bar{x}_2 & 0 & 0 & 0 & i_{23} & i_3 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{Bmatrix} \tag{5}$$

where $\bar{x}_2$ and $\bar{x}_3$ are offsets from the reference line of the cross-sectional mass centroid; and i_2, i_3, and i_{23} are cross-sectional mass moments and product of inertia. This equation may also be written as

$$\begin{Bmatrix} P \\ H \end{Bmatrix} = \begin{bmatrix} \mu\Delta & -\mu\tilde{\bar{x}} \\ \mu\tilde{\bar{x}} & I \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} \tag{6}$$

Closing the Formulation

As developed in Ref. 4, the formulation was closed by the use of a set of kinematical relations. This was undertaken by means of the introduction of displacement variables $u = \lfloor u_1 \;\; u_2 \;\; u_3 \rfloor^T$ and a suitable set of angular displacement measures. For the latter, Rodrigues parameters were previously used (see Ref. 4), but here the change in orientation is left in terms of the direction cosine matrix C. The kinematical relations are a set of generalized strain–displacement equations that relate γ and κ to u and C and a set of generalized velocity-displacement equations that relate V and Ω to u and C.

Generalized Strain–Displacement Equations

The generalized strain–displacement relations are of the form

$$\gamma = C(e_1 + u' + \tilde{k}u) - e_1 \tag{7}$$

$$\tilde{\kappa} = -C'C^T + C\tilde{k}C^T - \tilde{k} \tag{8}$$

where $\kappa = K - k$ and $k = \lfloor k_1 \;\; k_2 \;\; k_3 \rfloor^T$, and $\boldsymbol{k}(x_1)$ is the initial curvature/twist vector at x_1, such that $\boldsymbol{b}_i' = \boldsymbol{k} \times \boldsymbol{b}_i$. Thus, k_1 is the initial twist and k_2 and k_3 are the initial curvature measures.

Generalized Velocity–Displacement Equations

The generalized velocity–displacement relations are of a similar form, namely,

$$V = C\dot{u} \tag{9}$$

$$\tilde{\Omega} = -\dot{C}C^T \tag{10}$$

Note that the terms in the generalized velocity-displacement relations of Ref. 4 related to the frame motion have been set equal to zero. This means that the displacement and rotation measures are relative to an inertial frame and that the unit vectors $\boldsymbol{b}_i$ are fixed in that frame.

Derivation of Intrinsic Kinematical Equations

The derivation of intrinsic kinematical equations can be undertaken by careful elimination of all displacement and rotation variables from the generalized strain- and velocity-displacement equations. The detailed operations closely resemble the development of the transpositional relations in Ref. 4 and are only outlined here. Starting with differentiation of Ω with respect to x_1, one obtains

$$\tilde{\Omega}' = -\dot{C}'C^T - \dot{C}C'^T \tag{11}$$

Similarly, differentiation of $\tilde{\kappa}$ with respect to t yields

$$\dot{\tilde{\kappa}} = -\dot{C}'C^T - C'\dot{C}^T + \dot{C}\tilde{k}C^T + C\tilde{k}\dot{C}^T \tag{12}$$

These equations involve C', which can be expressed in terms of K using Eq. (8), and $\dot{C}$, which can be expressed in terms of Ω using Eq. (10). Both equations contain $\dot{C}'$, which can be eliminated with algebraic manipulation to yield a single relation between Ω' and $\dot{\kappa}$ that, when simplified, is given by

$$\Omega' + \tilde{K}\Omega = \dot{\kappa} \tag{13}$$

In a similar manner, V is differentiated with respect to x_1, yielding

$$V' = C'\dot{u} + C\dot{u}' \tag{14}$$

and γ is differentiated with respect to t, leading to

$$\dot{\gamma} = \dot{C}(e_1 + u' + \tilde{k}u) + C(\dot{u}' + \tilde{k}\dot{u}) \tag{15}$$

This time, $\dot{u} = C^T V$ and $(e_1 + u' + \tilde{k}u) = C^T(e_1 + \gamma)$ are used, and again Eqs. (8) and (10) are used to eliminate C' and $\dot{C}$, respectively. Now, one can eliminate $\dot{u}'$ and find a single relation between V' and $\dot{\gamma}$, namely,

$$V' + \tilde{K}V + (\tilde{e}_1 + \tilde{\gamma})\Omega = \dot{\gamma} \tag{16}$$

Eqs. (1), (4), (5), (13), and (16) constitute a closed formulation. The elegance of the formulation is striking, especially with regard to the similarity in structure of the left-hand sides of Eqs. (1a) and (13) and Eqs. (1b) and (16). Moreover, this formulation can actually be used in the solution of a variety of problems. For example, for situations in which the applied loads f and m and the boundary conditions on F, M, V, and Ω are independent of u and C, these equations allow the solution of nonlinear dynamics problems without finite rotation variables. Because these variables are frequently the source of the highest degree nonlinearities and a possible source of singularities or of the need for Lagrange multipliers or trigonometric functions, the ability to avoid finite rotation variables can be quite advantageous. Finally, this formulation leads to explicit expressions for two conservation laws, one of which is difficult to obtain in other ways. These laws may have practical applications in the development of computational algorithms. Before this aspect of the formulation is explored, an example is presented showing advantages of the formulation for stability problems involving nonconservative forces.

Example Showing Advantages of the Intrinsic Formulation

In this section, the utility of the fully intrinsic formulation will be addressed for problems involving nonconservative forces. The problem of Refs. 11 and 12 recently revisited,[13,14] provides an interesting illustration of the utility of the subject methodology for follower-force problems. In this problem, a cantilevered beam is loaded with

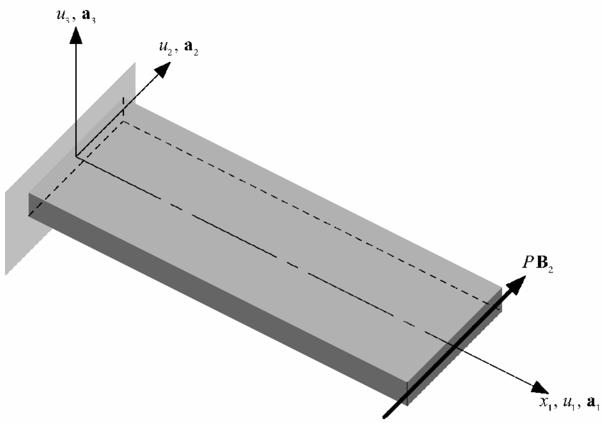

Fig. 2 Schematic of beam under transverse follower force.

a transverse follower force at its tip, as shown in Fig. 2. When a prismatic and isotropic beam is considered with the mass centroid coincident with the reference line and the principal axes of the cross section found along the $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$ directions, the constitutive law becomes

$$\begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{GA_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{GA_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{GJ} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{EI_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{EI_3} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{17}$$

and the cross-sectional generalized momentum–velocity relations are

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ H_1 \\ H_2 \\ H_3 \end{Bmatrix} = \begin{bmatrix} \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & i_2 + i_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & i_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & i_3 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{Bmatrix} \tag{18}$$

These relations, along with an exact linearization of Eqs. (1), (13), and (16), are used to produce governing equations for the stability of small motions about the static equilibrium state. Ignoring rotary inertia ($i_2 = i_3 = 0$) except in the torsional equation, considering infinite axial and shearing rigidities $[1/(EA) = 1/(GA_2) = 1/(GA_3) = 0]$, and letting $V(x_1, t) = \bar{V}(x_1) + \hat{V}(x_1, t)$ and similarly for all other variables, one obtains the following equations, linearized in the quantities with a carat superscript:

$$\begin{gathered} \hat{M}_1' + \bar{M}_3\hat{M}_2(1/EI_2 - 1/EI_3) - (i_2 + i_3)\dot{\hat{\Omega}}_1 = 0 \\ \hat{M}_2' - \bar{M}_3\hat{M}_1(1/GJ - 1/EI_3) - \hat{F}_3 = 0 \\ \hat{F}_3' + \bar{F}_2\hat{M}_1/GJ - \bar{F}_1\hat{M}_2/EI_2 - \mu\dot{\hat{V}}_3 = 0 \\ \hat{\Omega}_1' - \bar{M}_3\hat{\Omega}_2/EI_3 - \dot{\hat{M}}_1/GJ = 0 \\ \hat{\Omega}_2' + \bar{M}_3\hat{\Omega}_1/EI_3 - \dot{\hat{M}}_2/EI_2 = 0 \\ \hat{V}_3' + \hat{\Omega}_2 = 0 \end{gathered} \tag{19}$$

where the equilibrium state is governed by three first-order equations given by

$$\begin{gathered} \bar{F}_1' - \bar{M}_3\bar{F}_2/EI_3 = 0, \qquad \bar{F}_2' + \bar{M}_3\bar{F}_1/EI_3 = 0 \\ \bar{M}_3' + \bar{F}_2 = 0 \end{gathered} \tag{20}$$

The compactness and ease of derivation are noteworthy, as are the elegance and symmetry of the final equations. To appreciate the simplicity of the preceding formulation, one should compare it with the equations of Ref. 14. It is clear that the present formulation is considerably simpler. The simplicity of the present formulation for this problem stems from the boundary conditions' being independent of displacement and orientation variables, a property typical in follower-force problems.

Note that the present formulation will provide a simple alternative for some aeroelastic flutter analyses, although, in such cases, at least one measure of orientation may be needed. This normally can be done without the introduction of a full set of orientation angles or parameters because only one direction needs to be specified to calculate the angle of attack. The direction cosines for that direction, a 3×1 column matrix ϕ, are all one need introduce; their spatial derivatives can be related to the curvature measures as $\phi' + \tilde{K}\phi = 0$.

Conservation Laws for the Intrinsic Formulation

In this section, the conservation laws will be developed. To do so, an inverse form of Eq. (4) is helpful, which can be written as

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{bmatrix} R & S \\ S^T & T \end{bmatrix}^{-1} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \begin{Bmatrix} \left(\dfrac{\partial \mathcal{U}}{\partial \gamma}\right)^T \\ \left(\dfrac{\partial \mathcal{U}}{\partial \kappa}\right)^T \end{Bmatrix} \tag{21}$$

where the associated strain energy per unit length can be written as

$$\mathcal{U} = \frac{1}{2} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix}^T \begin{bmatrix} R & S \\ S^T & T \end{bmatrix}^{-1} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \tag{22}$$

Similarly, the momenta can be written as

$$\begin{Bmatrix} P \\ H \end{Bmatrix} = \begin{bmatrix} \mu\Delta & -\mu\tilde{\bar{x}} \\ \mu\tilde{\bar{x}} & I \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} = \begin{Bmatrix} \left(\dfrac{\partial \mathcal{T}}{\partial V}\right)^T \\ \left(\dfrac{\partial \mathcal{T}}{\partial \Omega}\right)^T \end{Bmatrix} \tag{23}$$

with the kinetic energy per unit length given by

$$\mathcal{T} = \frac{1}{2} \begin{Bmatrix} V \\ \Omega \end{Bmatrix}^T \begin{bmatrix} \mu\Delta & -\mu\tilde{\bar{x}} \\ \mu\tilde{\bar{x}} & I \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} \tag{24}$$

The approach for development of the conservation laws involves three steps:

1) Premultiply the equations of motion, Eqs. (1), by the transpose of either column matrix $\lfloor V^T \ \ \Omega^T \rfloor^T$ or $\lfloor (e_1 + \gamma)^T \ \ K^T \rfloor^T$, and integrate the result over space and time.

2) Integrate the resulting expression by parts to bring $\mathcal{U}$ and $\mathcal{T}$ into evidence.

3) Simplify the expressions using the space–time compatibility equations (13) and (16).

The first conservation law is derived by premultiplying the equations of motion by $\lfloor V^T \ \ \Omega^T \rfloor^T$, which leads to

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^{\ell} \{ & V^T(F' + \tilde{K}F + f - \dot{P} - \tilde{\Omega}P) + \Omega^T[M' + \tilde{K}M \\ & + (\tilde{e}_1 + \tilde{\gamma})F + m - \dot{H} - \tilde{\Omega}H - \tilde{V}P]\}\, dx_1\, dt = 0 \end{aligned} \tag{25}$$

Integration by parts yields

$$\int_{t_1}^{t_2}\int_0^\ell \left\{-V'^T F + V^T(\tilde{K}F + f) - \Omega'^T M + \Omega^T[\tilde{K}M + (\tilde{e}_1 + \tilde{\gamma})F + m]\right\} \mathrm{d}x_1\,\mathrm{d}t + \int_{t_1}^{t_2} (V^T F + \Omega^T M)\,\mathrm{d}t\Big|_0^\ell - \int_0^\ell \mathcal{T}\,\mathrm{d}x_1\Big|_{t_1}^{t_2} = 0 \tag{26}$$

After use of Eqs. (13) and (16) to eliminate V' and Ω' and a remarkable series of cancellations, one obtains

$$\int_{t_1}^{t_2}\int_0^\ell (V^T f + \Omega^T m)\,\mathrm{d}x_1\,\mathrm{d}t + \int_{t_1}^{t_2} (V^T F + \Omega^T M)\,\mathrm{d}t\Big|_0^\ell = \int_0^\ell (\mathcal{T} + \mathcal{U})\,\mathrm{d}x_1\Big|_{t_1}^{t_2} \tag{27}$$

The first term is clearly the work done by all applied forces and moments along the beam between times t_1 and t_2, and the second term is the work done by the forces and moments at the ends of the beam between times t_1 and t_2. The sum of these terms must equal the change in total mechanical energy, the term on the right-hand side. This conservation law is of the form that one should expect. Note that, both in this conservation law and the one to follow, ℓ can also be considered the length of a finite element.

Another conservation law can be found by again following the procedure just spelled out, except this time premultiplying by $\lfloor (e_1 + \gamma)^T \quad K^T \rfloor^T$. The result from step 1 is

$$\int_{t_1}^{t_2}\int_0^\ell \left\{(e_1+\gamma)^T(F' + \tilde{K}F + f - \dot{P} - \tilde{\Omega}P) + K^T[M' + \tilde{K}M + (\tilde{e}_1 + \tilde{\gamma})F + m - \dot{H} - \tilde{\Omega}H - \tilde{V}P]\right\} \mathrm{d}x_1\,\mathrm{d}t = 0 \tag{28}$$

Integration by parts and simplification subject to the restriction that k is constant yields

$$\int_{t_1}^{t_2}\int_0^\ell \left[\dot{\gamma}^T P + \dot{\kappa}^T H + (e_1+\gamma)^T(f - \tilde{\Omega}P) + K^T(m - \tilde{\Omega}H - \tilde{V}P)\right]\mathrm{d}x_1\,\mathrm{d}t + \int_{t_1}^{t_2}\left(e_1^T F + k^T M + \mathcal{U}\right)\mathrm{d}t\Big|_0^\ell - \int_0^\ell \left[(e_1+\gamma)^T P + K^T H\right]\mathrm{d}x_1\Big|_{t_1}^{t_2} = 0 \tag{29}$$

Eliminating $\dot{\gamma}$ and $\dot{\kappa}$ using Eqs. (13) and (16), and after cancellations similar to those in the derivation of Eq. (27), one finds

$$\int_{t_1}^{t_2}\int_0^\ell \left[(e_1+\gamma)^T f + K^T m\right]\mathrm{d}x_1\,\mathrm{d}t + \int_{t_1}^{t_2}\left(e_1^T F + k^T M + \mathcal{U} + \mathcal{T}\right)\mathrm{d}t\Big|_0^\ell = \int_0^\ell \left[(e_1+\gamma)^T P + K^T H\right]\mathrm{d}x_1\Big|_{t_1}^{t_2} \tag{30}$$

The first term is a sort of generalized impulse of all forces and moments along the beam between times t_1 and t_2, whereas the second at least appears to relate to the generalized impulse applied at the ends of the beam between times t_1 and t_2. The presence of the mechanical energy per unit length in this term, however, makes its identification problematic. The law states that the total of these must equal to the change in the generalized momentum expression on the right-hand side from t_1 to t_2.

Equations (27) and (30) are the conservation laws that were to be derived in this section. They can be used as benchmarks to evaluate the accuracy and self-consistency of any approximate solution, as well as criteria in the development of energy-preserving or -decaying discretization schemes for nonlinear structural dynamics. (See, for example, Refs. 15–18 and related work cited therein.)

Applications of the Conservation Laws

In this section, the utility of these conservation laws will be illustrated as applied to the discretized analysis of the statics of beams, the dynamics of rigid bodies, and the nonlinear dynamics of beams.

Kirchhoff Analogy

The Kirchhoff analogy is described by Love[1] in terms of a problem posed for rigid-body dynamics, the governing equations of which are exactly those of an inextensible beam undergoing static deformation. Although the former is normally posed as an initial-value problem, the latter is a two-point boundary-value problem. However, mathematically, both problems can be posed either way.

Nonlinear Statics of Beams

The nonlinear equations of equilibrium for an initially straight and untwisted beam ($k = 0$) that is rigid in extension and shear ($\gamma = 0$) and is subjected only to end loads ($f = m = 0$) are

$$F' + \tilde{\kappa}F = 0, \qquad M' + \tilde{\kappa}M + \tilde{e}_1 F = 0 \tag{31}$$

The constitutive law can be reduced to

$$M = T^{-1}\kappa = D\kappa \tag{32}$$

where $D = T^{-1}$ is taken as constant along the beam, and the orientation of each cross-sectional frame can be expressed in terms of C, where

$$C' = -\tilde{\kappa}C \tag{33}$$

The boundary conditions are for the end loads $F(0) = \hat{F}$ and $M(0) = \hat{M}$, where $\hat{F}$ and $\hat{M}$ are constant 3×1 column matrices. Here, $\hat{F}$ contains the measures in the $\boldsymbol{B}_i$ basis of a known applied force for which the measures in the $\boldsymbol{b}_i$ basis remain as constant as the beam deforms, that is, a dead force. The solution to the force equation is, thus, easily shown to satisfy the relation

$$F = CC(0)^T\hat{F} \tag{34}$$

which indicates, as expected, that the cross-sectional force resultant has a constant direction in the inertial frame. A clamped boundary at the other end yields a two-point boundary-value problem for which $C(\ell) = 0$.

The first conservation law, Eq. (27), is identically satisfied, whereas the second conservation law, Eq. (30), when specialized for this static example and for $f = m = k = 0$, requires that along the beam

$$e_1^T F + \mathcal{U} = \text{const} \tag{35}$$

This law and its extension to initially curved and twisted beams was discussed by Love[1] and referred to as an energy integral of the deformation.

In addition to the energy integral of the deformation, another integral can also be shown to exist. One first premultiplies the force equation by M^T and adds to the result the moment equation premultiplied by F^T. Because all other terms cancel, the result is that $(M^T F)' = 0$, or that, along the beam

$$M^T F = \text{const} \tag{36}$$

This integral reflects that the section force vector $\boldsymbol{F}$, which is constant along the beam, is everywhere perpendicular to changes in the section moment vector $\boldsymbol{M}$ along the beam. In other words, the component of section moment that is parallel to the section force remains constant along the beam. This integral can not be derived from either of the two conservation laws just derived.

It is interesting that some discretization schemes will satisfy these in each element, whereas others will only approach the satisfaction of them as the mesh is refined. The way in which a particular discretization scheme will satisfy these relations may have an effect on the quality of the scheme. To the best of the author's knowledge, however, this subject has not been investigated.

Dynamics of Rigid Bodies

Kirchhoff discovered that, regarding the prime as the time derivative, Eq. (31) is exactly Euler's dynamic equation for a rigid body that 1) is free to rotate about an inertially fixed point O, 2) is subjected only to a dead force $\boldsymbol{W} = W_i\boldsymbol{b}_i$ (for example, a gravitational force) passing through a point offset from O by $z\boldsymbol{B}_1$, where z and W_i are constants, 3) has an inertia matrix for O expressed in the body-fixed reference frame given by D, 4) has inertial angular velocity measures κ when expressed in the body-fixed frame, and 5) has measures of inertial angular momentum about O, expressed in the body frame basis, given by M.

To further elaborate on the loading, this means that in matrix notation $F = -zCW$, so that

$$F' = -zC'W = z\tilde{\kappa}CW = -\tilde{\kappa}F \tag{37}$$

as required by the first of Eqs. (31). The boundary conditions on force and moment for the beam are initial conditions on the applied load and angular momentum for the body. One may specify the direction cosines $C(0)$ and time march to an arbitrary time.

With regard to the integrals of deformation in Eqs. (35) and (36), for the rigid-body analog, the const. in both cases now means constant in time. Equation (35) now has the significance of conservation of total mechanical energy, which is the sum of the rotational kinetic energy of the body $\mathcal{U}$ about O and potential energy of the applied load $e_1^T F$. Equation (36) now stipulates conservation of the angular momentum of the body about a line passing through O and parallel to $\boldsymbol{W}$.

Discretization Schemes That Satisfy Specialized Conservation Laws

Given Kirchhoff's analogy, it really does not matter whether one considers discretization in space of the equations for the statics of beams or discretization in time of the equations for the dynamics of a rigid body. With regard to the static behavior of a beam, the most straightforward discretization is a simple finite difference scheme, such that, on the interior of a beam segment, F and M are approximated as averages of the nodal values on the left and right, denoted here by F_l and F_r for the force and M_l and M_r for the moment. Thus, Eqs. (31) can be written for one segment as

$$(F_r - F_l)/\Delta x + \tilde{\bar{\kappa}}\bar{F} = 0$$

$$(M_r - M_l)/\Delta x + \tilde{\bar{\kappa}}\bar{M} + \tilde{e}_1\bar{F} = 0 \tag{38}$$

where Δx is the segment length along x_1, and

$$\bar{F} = (F_r + F_l)/2, \qquad \bar{M} = (M_r + M_l)/2, \qquad \bar{\kappa} = T\bar{M} \tag{39}$$

It can be shown that this discretization scheme is identical to that described for the weakest possible mixed finite element formulation, which uses piecewise constant shape functions for all unknowns on the interior, discrete values of the unknowns at the ends, and piecewise linear test functions. This formulation is described in Ref. 4 and was used subsequently in several contexts.[5,7,8]

This scheme is second-order accurate along the beam. Moreover, for the rigid body, this scheme represents an implicit time-marching algorithm that is second-order accurate. In each case, it satisfies the appropriate conservation laws. That is, the time-marching algorithm for the rigid body is both energy and momentum conserving. This is important in the time domain to guarantee numerical stability.[15] Exactly what the satisfaction of these integrals implies in the space domain for the beam statics problem is not as clear. However, failure of a proposed solution to satisfy these integrals would cast doubt on its accuracy. Specifically, for finite element schemes, although its implications across a single element are less clear, these integrals should be satisfied at least in the limit of a fine mesh.

Discretization Scheme That Satisfies the Space–Time Conservation Laws

A far more interesting situation arises for the nonlinear, space–time treatment of beams. To generalize the preceding to a scheme that satisfies both conservation laws, the same scheme is applied to a rectagular element of the space–time domain such as that shown in Fig. 3. Here, the subscripts l, a, and r are retained for the left, average, and right values of any variable in its spatial x_1 variation, whereas s, m, and f refer to starting, mean, and final values for any variable in its temporal t variation. By the use of this scheme for a rectangular space–time element, a variable such as F has four nodal values: F_{ls}, F_{rs}, F_{lf}, and F_{lf}. The values along the edges are taken as the average of the nodal values on that edge such that

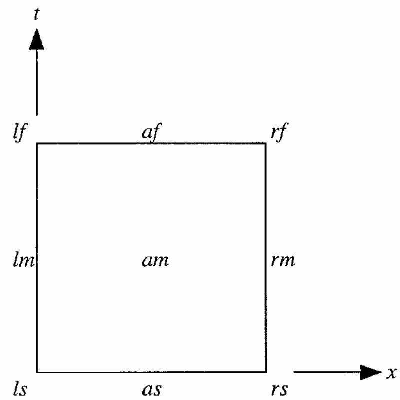

Fig. 3 Schematic of space–time finite element.

$$F_{lm} = (F_{ls} + F_{lf})/2, \qquad F_{rm} = (F_{rs} + F_{rf})/2$$

$$F_{as} = (F_{ls} + F_{rs})/2, \qquad F_{af} = (F_{lf} + F_{rf})/2 \tag{40}$$

Furthermore, the value on the interior of the element is

$$F_{am} = (F_{as} + F_{af})/2 = (F_{lm} + F_{rm})/2 = (F_{ls} + F_{lf} + F_{rs} + F_{rf})/4 \tag{41}$$

Such relations hold for variables M, V, and Ω as well. The intrinsic equations of motion along with the intrinsic kinematical equations are written in difference form as

$$(F_{rm} - F_{lm})/\Delta x + \tilde{K}_{am}F_{am} + f_{am} - (P_{af} - P_{as})/\Delta t - \tilde{\Omega}_{am}P_{am} = 0$$

$$(M_{rm} - M_{lm})/\Delta x + \tilde{K}_{am}M_{am} + (\tilde{e}_1 + \tilde{\gamma}_{am})F_{am} + m_{am} - (H_{af} - H_{as})/\Delta t - \tilde{\Omega}_{am}H_{am} - \tilde{V}_{am}P_{am} = 0$$

$$(V_{rm} - V_{lm})/\Delta x + \tilde{K}_{am}V_{am} + (\tilde{e}_1 + \tilde{\gamma}_{am})\Omega_{am} - (\gamma_{af} - \gamma_{as})/\Delta t = 0$$

$$(\Omega_{rm} - \Omega_{lm})/\Delta x + \tilde{K}_{am}\Omega_{am} - (\kappa_{af} - \kappa_{as})/\Delta t = 0 \tag{42}$$

where f, m, k, and the section constants are taken as constant over the space–time element and Δt is the time step.

For the special cases of longitudinal and torsional dynamics of rods, one can prove that this scheme satisfies both space–time conservation laws. This involves simply substituting the discretized equations of motion and kinematical equations directly into the conservation laws, which are satisfied identically. Unfortunately, for more general cases, because of nonlinear terms and complicated algebra, it does not appear to be possible to prove analytically that this discretization satisfies the conservation laws. However, the author investigated this by undertaking a numerical solution of the element equations. A wide variety of loading conditions and values for initial curvature and twist were substituted into numerical solutions for the general case of Eqs. (42) and were verified to satisfy both conservation laws to machine precision in every case tried. Of course, a complete study of the implications of this observation is beyond the scope of this paper.

As with the special cases, the failure of a converged result from any discretization scheme to satisfy these laws would certainly cast doubt on its accuracy. It is also possible that these laws could be used to indicate regions of high error in the space–time domain. Although the satisfaction of these laws at least appears to be necessary in some sense, there is no claim here that these laws are sufficient for accurate time simulation of the nonlinear dynamics of beams. High-frequency dissipation must be built in to any general-purpose scheme for time-marching of such equations.[17]

Conclusions

A formulation for the nonlinear dynamics of beams, based only on intrinsic equations, has been presented. The main advantage of this formulation appears to be that the nonlinearities are of a lower degree. This is because one need not introduce finite rotation variables unless the loads or boundary conditions depend on orientation. Even when they do, however, it is possible in some cases to circumvent the introduction of a full set of finite rotation variables. Advantages of this formulation are demonstrated through setting up a nonconservative stability problem. Also, the formulation naturally leads to two space–time conservation laws, one of which is the usual work–energy relation and the other of which is an apparently new generalized impulse–momentum relation. The work–energy relation is already known to facilitate the construction of energy-preserving and -decaying time-marching schemes in flexible multibody dynamics.

The similarity of the spatial and temporal operations to each other in all governing equations, as well as of those in the equations of motion to those in the kinematical equations, suggests a parallel approach to space–time discretization. By the use of the Kirchhoff analogy, it was shown that a time-discretization scheme for the nonlinear dynamics of rigid bodies can be used in space for the nonlinear static behavior of beams. Both of these schemes satisfy integrals of motion or deformation appropriate to the specialized problem, which for the dynamics problem exhibit conservation of energy and of one component of angular momentum. Of these two, only the conservation of energy can be derived from one of the space–time conservation laws, but it actually comes from a specialization of the new conservation law, not the work–energy relation. Remarkably, a straightforward generalization of these discretization schemes leads to a space–time discretization scheme that satisfies both space–time conservation laws. Evidently, both space–time conservation laws are potentially useful in checking numerical algorithms and indicating error. Additional uses for them will be considered in future work.

Acknowledgments

The author gratefully acknowledges hours of technical discussion with his esteemed colleague Olivier A. Bauchau, who suggested the form of the space–time discretization scheme presented herein. Critical remarks by Mark V. Fulton, Mayuresh J. Patil, and Gene C. Ruzicka were quite helpful in refining the presentation.

References

[1]Love, A. E. H., *Mathematical Theory of Elasticity*, 4th ed., Dover, New York, 1944, Articles 260, 261.

[2]Reissner, E., "On One-Dimensional Large-Displacement Finite-Strain Beam Theory," *Studies in Applied Mathematics*, Vol. 52, No. 2, 1973, pp. 87–95.

[3]Borri, M., and Mantegazza, P., "Some Contributions on Structural and Dynamic Modeling of Helicopter Rotor Blades," *l'Aerotecnica Missili e Spazio*, Vol. 64, No. 9, 1985, pp. 143–154.

[4]Hodges, D. H., "A Mixed Variational Formulation Based on Exact Intrinsic Equations for Dynamics of Moving Beams," *International Journal of Solids and Structures*, Vol. 26, No. 11, 1990, pp. 1253–1273.

[5]Patil, M. J., Hodges, D. H., and Cesnik, C. E. S., "Nonlinear Aeroelastic Analysis of Complete Aircraft in Subsonic Flow," *Journal of Aircraft*, Vol. 37, No. 5, 2000, pp. 753–760.

[6]Hodges, D. H., Patil, M. J., and Chae, S., "Effect of Thrust on Bending–Torsion Flutter of Wings," *Journal of Aircraft*, Vol. 39, No. 2, 2002, pp. 371–376.

[7]Hodges, D. H., Shang, X., and Cesnik, C. E. S., "Finite Element Solution of Nonlinear Intrinsic Equations for Curved Composite Beams," *Journal of the American Helicopter Society*, Vol. 41, No. 4, 1996, pp. 313–321.

[8]Shang, X., Hodges, D. H., and Peters, D. A., "Aeroelastic Stability of Composite Hingeless Rotors in Hover with Finite-State Unsteady Aerodynamics," *Journal of the American Helicopter Society*, Vol. 44, No. 3, 1999, pp. 206–221.

[9]Borri, M., Ghiringhelli, G. L., and Merlini, T., "Linear Analysis of Naturally Curved and Twisted Anisotropic Beams," *Composites Engineering*, Vol. 2, No. 5–7, 1992, pp. 433–456.

[10]Yu, W., Hodges, D. H., Volovoi, V. V., and Cesnik, C. E. S., "On Timoshenko-Like Modeling of Initially Curved and Twisted Composite Beams," *International Journal of Solids and Structures*, Vol. 39, No. 19, 2002, pp. 5101–5121.

[11]Barsoum, R. S., "Finite Element Method Applied to the Problem of Stability of a Nonconservative System," *International Journal for Numerical Methods in Engineering*, Vol. 3, 1971, pp. 63–87.

[12]Wohlhart, K., "Dynamische Kippstabilität eines Plattenstreifens unter Folgelast," *Zeitschrift für Flugwissenschaften*, Vol. 19, No. 7, 1971, pp. 291–298.

[13]Hodges, D. H., "Lateral–Torsional Flutter of a Deep Cantilever Loaded by a Lateral Follower Force at the Tip," *Journal of Sound and Vibration*, Vol. 247, No. 1, 2001, pp. 175–183.

[14]Detinko, F. M., "Some Phenomena for Lateral Flutter of Beams Under Follower Load," *International Journal of Solids and Structures*, Vol. 39, No. 2, 2002, pp. 341–350.

[15]Simo, J. C., and Wong, K. K., "Unconditionally Stable Algorithms for Rigid Body Dynamics That Exactly Preserve Energy and Momentum," *International Journal for Numerical Methods in Engineering*, Vol. 31, 1991, pp. 19–52.

[16]Simo, J. C., and Tarnow, N., "The Discrete Energy-Momentum Method. Conserving Algorithms for Non-Linear Dynamics," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 43, 1992, pp. 757–792.

[17]Bauchau, O. A., and Theron, N. J., "Energy Decaying Schemes for Nonlinear Beam Models," *Computer Methods in Applied Mechanics and Engineering*, Vol. 134, 1996, pp. 37–56.

[18]Bauchau, O. A., and Bottasso, C. L., "On the Design of Energy Preserving and Decaying Schemes for Flexible, Non-Linear Multi-Body Systems," *Computer Methods in Applied Mechanics and Engineering*, Vol. 169, 1999, pp. 61–79.

C. Pierre
Associate Editor